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A TRANSPORT EQUATION DESCRIPTION OF NON-LINEAR OCEAN SURFACE WAVE INTERACTIONS

Kenneth M. Watson, et al

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ABSTRACT

The evolution of the power spectrum of surface gravity waves is described by means of a transport equation. The effects of a slowly varying, prescribed ocean current and nonlinear wave-wave interactions are included. A definition due to Wigner of a localized power spectrum is used to derive the transport equation from the dynamical equations describing surface wave motion.

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I. Introduction

In this paper we present a derivation of a transport equation to describe the evolution of the power spectrum of surface waves in the deep ocean. Our transport equation is obtained directly from the dynamical description of surface gravity waves and is to be contrasted with more empirical formulations (Barnett, 1968; Thomson and West, 1973). Included will be the effect of a slowly varying, prescribed current and the influence of nonlinear wave-wave interactions. The dynamic equations are in mode coupled form and are obtained by expanding the surface displacement and velocity potential in Fourier series (Hasselmann, 1961, 1963; West, et al., 1974). Our treatment begins, as does that of Hasselmann (1961, 1963), with the time dependent equations for these Fourier coefficients. Using the definition of a local power spectrum due to Wigner (1932), a transport equation is obtained that has interaction terms of lower order than those obtained by Hasselmann (1961, 1963).

II. The Eigenmode Equations for Surface Waves

The difficulty with a strictly phenomenological construction of a spectral transport equation (Thomson and West, 1973) is the inherent uncertainty in modeling the individual interactive mechanisms. Also, because of the nonlinear nature of the interaction process, superposition of separate mechanisms is not formally justified, although it may be pragmatically. To obviate these complications we use the dynamic equations for the interaction of surface waves to construct a transport equation. We are concerned with a description of the evolving spectrum of surface gravity waves due both to the nonlinear interactions among these waves and to the interaction with a prescribed oceanic current. Such a current might represent tidal currents at the mouth of a bay or estuary, perhaps wind-driven currents, etc. We assume the characteristic distances over which this current varies, as well as the ocean depth, to be very large compared to the wavelength of the surface waves being studied.

The basis for our development will be a set of eigenmode equations similar to ones published previously (West, et al., 1974). Incompressible, irrotational flow is assumed. The fluid velocity is thus expressed as the gradient of a velocity potential Φ,

$$\Phi = \phi + \hat{\Phi} \quad . \tag{1}$$

In Eq. (1) the velocity potential is given by a linear superposition of the short wavelength, high frequency surface waves represented by ϕ and the long wavelength, slowly varying prescribed current represented by $\hat{\Phi}$.

The undisturbed ocean surface is assumed to coincide with the plane $z = 0$ of a rectangular coordinate system. The z axis is directed upward and the (x,y) plane lies in this surface. The horizontal flow associated with the prescribed current is

$$\underset{\sim}{U}(\underset{\sim}{x},t) = \nabla_s \hat{\Phi} \qquad , \qquad z = 0 \; , \tag{2}$$

where ∇_s is the gradient operator acting in the horizontal plane, $\underset{\sim}{x} = (x,y)$ is a vector in this plane, and we assume $\hat{\Phi}$ to vary slowly with z so we can evaluate $\underset{\sim}{U}$ at $z = 0$. The vertical flow of the current $\left|\frac{\partial \hat{\Phi}}{\partial z}\right|$ is assumed to be very small compared with $|\underset{\sim}{U}|$, and the surface displacement $H(\underset{\sim}{x},t)$ due to this flow is also assumed to be correspondingly small.

When surface waves are present the equation of the sea surface is of the form,

$$z = H(\underset{\sim}{x},t) + \zeta(\underset{\sim}{x},t) \tag{3}$$

where $\zeta(\underset{\sim}{x},t)$ represents the short wavelength vertical displacement due to surface gravity waves. Because $H(\underset{\sim}{x},t)$ is considered to be very small, in our discussion we shall replace Eq. (3) by the equation

$$z = \zeta(\underset{\sim}{x},t). \tag{4}$$

(The self-consistent hydrodynamic implications of the assumptions made about $H(\underset{\sim}{x},t)$ and $\frac{\partial\hat{\phi}}{\partial z}$ are discussed in Appendix A.)

In West, et al., (1974) the velocity potential Φ and vertical displacement ζ were represented as discrete Fourier series in a rectangular ocean of large area A_o. Time-dependent equations were obtained for these Fourier coefficients, similar to those previously obtained by other authors (Phillips, 1960; Benney, 1962; Hasselmann, 1961, 1963), and numerical integration of these equations was described. For our present application, it is desirable to introduce a somewhat modified modal analysis. The velocity potential

$$\phi_s(\underset{\sim}{x},t) \equiv \phi(\underset{\sim}{x},z,t) \quad , \quad \text{at } z = \zeta(\underset{\sim}{x},t) \quad ,$$

is used instead of ϕ_o, defined as ϕ on the plane $z = 0$. The difficulty with the use of ϕ_o is that the shorter wavelengths may undergo many e-foldings of attenuation between the true ocean surface and the plane $z = 0$.

We represent the flow by the complex amplitude $Z(\underset{\sim}{x},t)$, defined by the equations

$$\phi_s = V_x(Z+Z^*)/2 \quad ,$$

$$\zeta = i(Z-Z^*)/2 \quad . \tag{5}$$

Here V_x is the "velocity operator" (g is the acceleration of gravity)

$$V_x \equiv (g/\textcircled{\#})^{\frac{1}{2}} \equiv \omega_x/\textcircled{\#} \quad ,$$

$$\textcircled{\#} \equiv (-\nabla_s^2)^{\frac{1}{2}} \quad . \tag{6}$$

These quantities are assumed to operate on functions expressed as Fourier series, for which the proper operation is self-evident. For example, we assume Z to be a function defined in a rectangular ocean of area A_o and write

$$Z(\underset{\sim}{x},t) = \sum_{\underset{\sim}{k}} A(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \quad , \tag{7}$$

where the Fourier coefficients $A(\underset{\sim}{k})$ are time dependent. Thus,

$$V_x Z = \sum_{\underset{\sim}{k}} V_k\, A(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \quad ,$$

where

$$V_k \equiv (g/k)^{\frac{1}{2}} \equiv \omega_k/k \tag{8}$$

is the phase velocity of a small amplitude surface gravity wave of wavenumber k in deep water. The corresponding angular frequency is $\omega_k = (gk)^{\frac{1}{2}}$.

In our rectangular two-dimensional space representing the quiescent ocean surface, the Fourier exponentials satisfy the relations

$$A_o^{-1} \int d^2x \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x}) = \delta_{\underset{\sim}{k}} \quad ,$$

$$A_o^{-1} \sum_{\underset{\sim}{k}} \exp(i\underset{\sim}{k} \cdot \underset{\sim}{x}) = \delta(\underset{\sim}{x}) \tag{9}$$

where $\delta_{\underset{\sim}{k}}$ is the Kronecker and $\delta(\underset{\sim}{x})$ the Dirac delta function.

The prescribed current is given the Fourier representation

$$\underset{\sim}{U}(\underset{\sim}{x},t) = \sum_{\underset{\sim}{K}} \underset{\sim}{U}(\underset{\sim}{K}) \cos(\underset{\sim}{K} \cdot \underset{\sim}{x} - \Omega_K t) \tag{10}$$

where Ω_K is some (presently) unspecified function of $|\underset{\sim}{K}|$ and the vector mode amplitudes $\underset{\sim}{U}(\underset{\sim}{K})$ are also unspecified.

It is straightforward to obtain from the fluid dynamic equations the time-dependent differential equations satisfied by the Fourier amplitudes $A(\underset{\sim}{k})$ of Eq. (7). This procedure is outlined in Appendix B for the case that there is no prescribed current, or U = 0. The additional terms required to account for this current are obtained in Appendix A in an approximation that keeps only terms linear in the surface wave amplitudes. The resulting equations are [here $\dot{A} \equiv dA/dt$]

$$\dot{A}(\underset{\sim}{k}) + i\omega_k A(\underset{\sim}{k}) = T_W(A) + T_U(A) + T_2(A) + T_3(A) + \ldots . \quad (11)$$

In obtaining Eqs. (11) we have neglected surface tension and have supposed the ocean to be much deeper than the longest wavelengths considered. We must therefore set equal to zero those amplitudes corresponding to capillary waves or corresponding to wavelengths comparable to or greater than the depth, e.g., in Eq. (11), T_W models the effect of wind and viscosity. Based on a model of Miles (1957, 1960) (see, also, Phillips, 1969), we adopt for this the simple linear expression [an explicit derivation was given in West, et al. (1974)]

$$T_W(A) = \left[\underset{\sim}{\alpha} \cdot \underset{\sim}{k}/(2V_k) - \nu k^2\right] A(\underset{\sim}{k}) \quad . \qquad (12)$$

Here $\underset{\sim}{\alpha}$ is a vector parallel to the wind direction and having a magnitude dependent on the wind speed.

The quantity T_U describes the coupling to the prescribed current. As obtained in Appendix A, this has the form

$$T_U = -i \sum_{\underset{\sim}{K}} \Big[C^{(-)}(\underset{\sim}{k},\underset{\sim}{K})\, A(\underset{\sim}{k}-\underset{\sim}{K})\, \exp(-i\Omega_K t) + C^{(+)}(\underset{\sim}{k},\underset{\sim}{K})\, A(\underset{\sim}{k}+\underset{\sim}{K})\, \exp(i\Omega_K t) \Big] , \tag{13}$$

where

$$C^{(\pm)}(\underset{\sim}{k},\underset{\sim}{K}) = \tfrac{1}{4}\, \underset{\sim}{U}(\underset{\sim}{K}) \cdot \left[\underset{\sim}{k} + (\underset{\sim}{k} \pm \underset{\sim}{K}) \frac{\omega_k}{\omega_{|\underset{\sim}{k} \pm \underset{\sim}{K}|}} \right] . \tag{14}$$

The term T_2 in Eq. (11) represents the nonlinear wave-wave interaction of $\mathcal{O}(A^2)$. This is derived in Appendix B and has the form

$$T_2(A) = \sum_{\underset{\sim}{\ell},\underset{\sim}{p}} \delta_{\underset{\sim}{k}-\underset{\sim}{\ell}-\underset{\sim}{p}} \Big[\Gamma^{\underset{\sim}{k}}_{\underset{\sim}{\ell},\underset{\sim}{p}}\, A(\underset{\sim}{\ell})\, A(\underset{\sim}{p}) + \Gamma^{\underset{\sim}{k},-\underset{\sim}{p}}_{\underset{\sim}{\ell}}\, A(\underset{\sim}{\ell})\, A^*(-\underset{\sim}{p}) + \Gamma^{\underset{\sim}{k},-\underset{\sim}{p},-\underset{\sim}{\ell}}\, A^*(-\underset{\sim}{\ell})\, A^*(-\underset{\sim}{p}) \Big] . \tag{15}$$

The explicit expressions for the coefficients Γ are given in Eq. (B.7) of Appendix B.

Finally, the term T_3 describes nonlinear wave interactions of $\mathcal{O}(A^3)$. This is also shown in Appendix B to have the form

$$T_3(A) = \frac{i}{4} \sum_{\underset{\sim}{\ell}, \underset{\sim}{p}, \underset{\sim}{n}} \delta_{\underset{\sim}{k}+\underset{\sim}{n}-\underset{\sim}{\ell}-\underset{\sim}{p}} \Big[\Gamma_{\underset{\sim}{\ell},\underset{\sim}{p}}^{\underset{\sim}{k},\underset{\sim}{n}} A(\underset{\sim}{\ell})\, A(\underset{\sim}{p})\, A^*(\underset{\sim}{n})$$

$$+ \Gamma_{\underset{\sim}{\ell},\underset{\sim}{p},-\underset{\sim}{n}}^{\underset{\sim}{k}} A(\underset{\sim}{\ell})\, A(\underset{\sim}{p})\, A(-\underset{\sim}{n}) + \Gamma_{\underset{\sim}{\ell}}^{\underset{\sim}{k},-\underset{\sim}{p},\underset{\sim}{n}} A(\underset{\sim}{\ell})\, A^*(-\underset{\sim}{p})\, A^*(\underset{\sim}{n})$$

$$+ \Gamma^{\underset{\sim}{k},-\underset{\sim}{p},-\underset{\sim}{\ell},\underset{\sim}{n}} A^*(-\underset{\sim}{\ell})\, A^*(-\underset{\sim}{p})\, A^*(\underset{\sim}{n}) \Big] \quad . \qquad (16)$$

Of the above coefficients, only $\Gamma_{\underset{\sim}{\ell},\underset{\sim}{p}}^{\underset{\sim}{k},\underset{\sim}{n}}$ will be needed in this paper. This is given in Eq. (B.8) of Appendix B. The remaining terms in Eq. (16) tend to have rapidly oscillating exponentials and are not expected to contribute significantly to transfer of excitation between modes in Eq. (11).

The final dots in Eq. (11) indicate that we have included only terms to $\mathcal{O}(A^3)$ in the interaction. That is, we suppose the amplitudes to be sufficiently small that terms of $\mathcal{O}(A^4)$ and higher may be neglected.

III. Correlation Functions and the Power Spectrum of the Wave Amplitude

We wish to construct spectral transport equations from Eq. (11). To do this we first introduce an ensemble average, indicated as

$$"\langle \cdots \rangle" ,$$

over many observations of the sea state. In this way we can construct a hierarchy of average quantities such as

$$\langle A(\underset{\sim}{k})\rangle , \langle A(\underset{\sim}{k})\ A(\underset{\sim}{\ell})\rangle , \text{ etc.}$$

Using Eq. (11), we then obtain a corresponding hierarchy of coupled equations for these quantities. To close this set we need a statistical postulate to express the higher order correlation functions in terms of lower order correlation functions.

The postulate which we adopt is that

$$\langle A(\underset{\sim}{\ell})\ A(\underset{\sim}{p})\ A^*(\underset{\sim}{k})\ A^*(\underset{\sim}{n})\rangle = \langle A(\underset{\sim}{\ell})\ A^*(\underset{\sim}{k})\rangle \langle A(\underset{\sim}{p})\ A^*(\underset{\sim}{n})\rangle + \langle A(\underset{\sim}{\ell})\ A^*(\underset{\sim}{n})\rangle \langle A(\underset{\sim}{p})\ A^*(\underset{\sim}{k})\rangle , \qquad (17)$$

with <u>all</u> <u>other</u> fourth-order correlation functions vanishing.

We shall also neglect as small all correlation products involving more than four factors of A and A^*.

The above postulate permits us to use Eq. (11) to express the averages of products of one, two, and three A's in terms of the $\langle A(\underset{\sim}{\ell})\, A^*(\underset{\sim}{n})\rangle$ and to also obtain an equation to determine this second order correlation function.

The postulate (17) has arisen in a variety of guises in other applications of statistical mechanics, perhaps the earliest being Boltzmann's assumption of "molecular chaos". It is clearly only an approximation. In the remainder of this paper we shall accept this postulate and shall not attempt to assess its validity here.

At this point it is convenient (but not necessary) to remove the term T_2 from Eq. (11) with a transformation on the Fourier amplitudes. We write

$$A(\underset{\sim}{k}) = a(\underset{\sim}{k}) + G(\underset{\sim}{k}) \quad , \tag{18}$$

where G satisfies the equation

$$\dot{G}(\underset{\sim}{k}) + i\omega_k\, G(\underset{\sim}{k}) = T_2(a) \quad . \tag{19}$$

This equation may be formally integrated to give

$$G(\underset{\sim}{k}) = e^{-i\omega_k t} \int^{t} e^{i\omega_k t'} T_2(a)\, dt' \quad . \tag{20}$$

Thus G is of $\mathcal{O}(a^2)$. The difference

$$T_2'(a) \equiv T_2(A) - T_2(a) \quad ,$$

expressed as a functional of the a's is of $\mathcal{O}(a^3)$. This lets us finally re-write Eq. (11) in the form

$$\dot{a}(\underset{\sim}{k}) + i\omega_k\, a(\underset{\sim}{k}) = T_W(a) + T_U(a) + T_3'(a) \quad , \tag{21}$$

where

$$T_3'(a) \equiv T_3(a) + T_2'(a) \tag{22}$$

and we have dropped terms of order higher than $\mathcal{O}(a^3)$. We have also dropped the higher order terms in T_W and T_U, which is consistent with our use of only simple linear models for these.

Equation (21) contains only terms with an odd number of factors of the a's. Thus, it is consistent with this equation to require that the average of any product with an odd number of a-factors vanish. We have then, for example,

$$\langle a(\underset{\sim}{k})\rangle = \langle a(\underset{\sim}{k})\; a(\underset{\sim}{\ell})\; a^*(\underset{\sim}{n})\rangle = 0 \quad , \tag{23}$$

etc. The corresponding averages of the A's, obtained with the use of Eq. (18), are not expected to vanish. Since we are neglecting higher than fourth order correlation functions, we may simply replace the A's by the quantities a in Eq. (17).

We shall formally suppose that T_W, T_U, and T_3 are of the same order of smallness in Eq. (21). This permits us to evaluate Eq. (20) in a simple approximation, writing

$$a(\underset{\sim}{k}) = q(\underset{\sim}{k})\, e^{-i\omega_k t}$$

and considering the time variation of the q's to be very slow. Then, we obtain

$$G(\underset{\sim}{k}) \cong i \sum_{\underset{\sim}{\ell},\underset{\sim}{p}} \delta_{\underset{\sim}{k}-\underset{\sim}{\ell}-\underset{\sim}{p}} \left[\frac{\Gamma^{\underset{\sim}{k}}_{\underset{\sim}{\ell}\underset{\sim}{p}}\, a(\underset{\sim}{\ell})\, a(\underset{\sim}{p})}{\omega_\ell + \omega_p - \omega_k} + \frac{\Gamma^{\underset{\sim}{k},-\underset{\sim}{p}}_{\underset{\sim}{\ell}}\, a(\underset{\sim}{\ell})\, a^*(-\underset{\sim}{p})}{\omega_\ell - \omega_p - \omega_k} - \frac{\Gamma^{\underset{\sim}{k},-\underset{\sim}{\ell},-\underset{\sim}{p}}\, a^*(-\underset{\sim}{\ell})\, a^*(-\underset{\sim}{p})}{\omega_\ell + \omega_p + \omega_k} \right]. \quad (24)$$

Substitution into Eq. (22) lets us finally write

$$T_3'(a) = \frac{i}{4} \sum_{\underset{\sim}{\ell},\underset{\sim}{p},\underset{\sim}{n}} \delta_{\underset{\sim}{k}+\underset{\sim}{n}-\underset{\sim}{\ell}-\underset{\sim}{p}}\, C^{\underset{\sim}{k},\underset{\sim}{n}}_{\underset{\sim}{\ell},\underset{\sim}{p}}\, a(\underset{\sim}{\ell})\, a(\underset{\sim}{p})\, a^*(\underset{\sim}{n}) + \text{terms not needed.} \quad (25)$$

The "terms not needed" here have the form of the final three terms in Eq. (16). They have rapidly oscillating exponentials and will not contribute to the calculation of the correlation functions $\langle a(\underset{\sim}{\ell})\, a^*(\underset{\sim}{k})\rangle$, according to our statistical postulate (17) and its accompanying postulates. Were we to evaluate $\langle a(\underset{\sim}{\ell})\, a(\underset{\sim}{p})\rangle$, on the other hand, we would require some of these other terms in T_3'. The coefficient $C^{\underset{\sim}{k},\underset{\sim}{n}}_{\underset{\sim}{\ell},\underset{\sim}{p}}$ is given in Appendix C.

Following Wigner (1932) we now introduce the power spectrum of the a's with the definition

$$F(\underset{\sim}{x},\underset{\sim}{k}) = \tfrac{1}{2}\sum_{\underset{\sim}{\rho}} e^{i\underset{\sim}{\rho}\cdot\underset{\sim}{x}} \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2)\rangle$$

$$= (2A_o)^{-1}\int d^2r\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \langle \zeta(\underset{\sim}{x}+\underset{\sim}{r}/2)\; \zeta^*(\underset{\sim}{x}-\underset{\sim}{r}/2)\rangle \quad . \qquad (26)$$

Here

$$\zeta(\underset{\sim}{x},t) \equiv \sum_{\underset{\sim}{k}} a(\underset{\sim}{k})\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \quad , \qquad (27)$$

which differes from the quantity (7) by terms of $\mathcal{O}(a^2)$. Using Eq. (5), we see that

$$\sum_{\underset{\sim}{k}} F(\underset{\sim}{x},\underset{\sim}{k}) = \langle \zeta^2(\underset{\sim}{x})\rangle + \text{terms of } \mathcal{O}(a^4). \qquad (28$$

In the next section we shall use Eq. (21) and the statistical postulate (17) to obtain an equation for $F(\underset{\sim}{x},\underset{\sim}{k})$. The other correlation functions, $\langle a(\underset{\sim}{\ell})\, a(\underset{\sim}{p})\rangle$ and its complex conjugate, may then be evaluated in terms of the F's $\left[\text{of } \mathcal{O}(F^2)\right]$ by an argument similar to that by which Eq. (24) was obtained. Using the above, power spectra for wave energy or wave amplitude can be constructed. We may then consider $F(\underset{\sim}{x},\underset{\sim}{k})$ to represent an approximation to the power spectrum for wave amplitude. The precise power spectrum for wave amplitude will contain additional terms of $\mathcal{O}(F^2)$, which can be readily evaluated as just described.

For most applications it is convenient to change from discrete to continuum normalization by replacing the sum over discrete wavenumbers by integrals with the substitution.

$$\sum_{\underset{\sim}{k}} \rightarrow \frac{A_o}{(2\pi)^2}\int d^2k \quad . \tag{29}$$

This allows us to define

$$\Psi(\underset{\sim}{x},\underset{\sim}{k}) \equiv \frac{A_o}{(2\pi)^2} F(\underset{\sim}{x},\underset{\sim}{k}) \tag{30}$$

with the normalization

$$\int d^2k\ \Psi(\underset{\sim}{x},\underset{\sim}{k}) \cong \langle \zeta^2(\underset{\sim}{x})\rangle \tag{31}$$

in the approximation of Eq. (28). The function $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ defined by Eq. (30) provides a generalization of the spectral function $\Psi(\underset{\sim}{k})$ described by Phillips (1969) for a spatially homogeneous ocean.

In practice, the Wigner spectral function [Eq. (30)] is useful only if $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ varies <u>very</u> slowly over distances comparable to k^{-1} for all k of interest. For oceanic applications this condition is usually well satisfied except near physical discontinuities (such as the shore). We thus introduce a characteristic distance W over which $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ varies appreciably and assume that

$$k \gg W^{-1} \tag{32}$$

for those k of interest*. Referring back to Eq. (26) we see that Eq. (32) implies that

$$\left\langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2\right\rangle \approx 0 \tag{33}$$

for $|\underset{\sim}{\rho}| \gg W^{-1}$.

The spectrum of energy per unit area is, correct to second order in the a's,

$$E(\underset{\sim}{x},\underset{\sim}{k}) = \rho_o g\; \Psi(\underset{\sim}{x},\underset{\sim}{k})$$

where ρ_o is the sea water density.

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*This, for example, implies that k >> K in Eq. (13).

If we write $a(\underset{\sim}{k},t)$ to indicate the explicit time dependence of the $a(\underset{\sim}{k})$'s, we may express the spectral distribution for wavenumber and frequency in the form

$$\Psi(\underset{\sim}{x},\underset{\sim}{k},t,\omega) = \left\{A_o/\left[2(2\pi)^3\right]\right\} \sum_{\underset{\sim}{\rho}} \int d\tau \, \exp(i\underset{\sim}{\rho}\cdot\underset{\sim}{x} + i\omega\tau)$$
$$\times \left\langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2,\ t+\tau/2)\ a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2,\ t-\tau/2)\right\rangle \quad .$$

The equation satisfied by this quantity is more complicated than that derived in the next section for $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ and will not be described here.

IV. The Spectral Transport Equation

In the preceeding section we introduced the spectral function in terms of an ensemble average over products of the eigenmode amplitudes. In Appendix A we presented the dynamic equations for the mutual interaction of these surface eigenmodes and their interaction with a prescribed surface current. In this section we synthesize these approaches to construct an equation for the evolution of the spectral function $\Psi(\underset{\sim}{x},\underset{\sim}{k})$. To obtain this equation we differentiate the first form of Eq. (26) with respect to time;

$$\frac{\partial F(\underset{\sim}{x},\underset{\sim}{k})}{\partial t} = \tfrac{1}{2} \sum_{\underset{\sim}{\ell}} \left[\left\langle \dot{a}(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^{*}(\underset{\sim}{k}-\underset{\sim}{\rho}/2) \right\rangle + \left\langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; \dot{a}^{*}(\underset{\sim}{k}-\underset{\sim}{\rho}/2) \right\rangle \right]$$

$$\times \exp(i\underset{\sim}{\rho}\cdot\underset{\sim}{x}) \quad . \qquad (34)$$

The time derivative of the complex amplitude $a(\underset{\sim}{k})$ can be eliminated from Eq. (34) by substitution from Eq. (21). We then obtain on the right-hand side of Eq. (34) a sum of terms involving correlation functions such as

$$\left\langle a(\underset{\sim}{k})\; a^{*}(\underset{\sim}{\ell}) \right\rangle \qquad (35a)$$

and

$$\langle a(\underset{\sim}{p})\; a(\underset{\sim}{\ell})\; a^*(\underset{\sim}{n})\; a^*(\underset{\sim}{k})\rangle \quad . \tag{35b}$$

The spectral function $F(\underset{\sim}{x},\underset{\sim}{k})$ is of course just a Fourier transformed version of Eq. (35a).

The classification of correlation functions given by Eq. (35) suggest that we rewrite Eq. (34) in the form

$$\frac{\partial F(\underset{\sim}{x},\underset{\sim}{k})}{\partial t} = T_a + T_b \quad . \tag{36}$$

We consider first the rather simple T_a term. This is

$$\begin{aligned} T_a = -\frac{i}{2}\sum_{\underset{\sim}{\rho}} \Bigg\{ & \left[\omega_{|\underset{\sim}{k}+\underset{\sim}{\rho}/2|} - \omega_{|\underset{\sim}{k}-\underset{\sim}{\rho}/2|} + \alpha(\underset{\sim}{k}+\underset{\sim}{\rho}/2) + \alpha(\underset{\sim}{k}-\underset{\sim}{\rho}/2)\right] \\ & \times \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2)\rangle \\ & + \sum_{\underset{\sim}{K}} \Big[C^{(-)}(\underset{\sim}{k}+\underset{\sim}{\rho}/2,\underset{\sim}{K})\, \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2-\underset{\sim}{K})\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2)\rangle \exp(-i\Omega_K t) \\ & + C^{(+)}(\underset{\sim}{k}+\underset{\sim}{\rho}/2,\underset{\sim}{K})\, \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2+\underset{\sim}{K})\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2)\rangle \exp(i\Omega_K t) \\ & - C^{(-)}(\underset{\sim}{k}-\underset{\sim}{\rho}/2,\underset{\sim}{K})\, \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2-\underset{\sim}{K})\rangle \exp(i\Omega_K t) \\ & - C^{(+)}(\underset{\sim}{k}-\underset{\sim}{\rho}/2,\underset{\sim}{K})\, \langle a(\underset{\sim}{k}+\underset{\sim}{\rho}/2)\; a^*(\underset{\sim}{k}-\underset{\sim}{\rho}/2+\underset{\sim}{K})\rangle \exp(-i\Omega_K t)\Big]\Bigg\} \exp(i\underset{\sim}{\rho}\cdot\underset{\sim}{x}) \end{aligned} \tag{37}$$

where $\alpha(\underset{\sim}{k}) = i\left(\underset{\sim}{\alpha}\cdot\underset{\sim}{k}/2v_k - \nu k^2\right)$.

Because of the variation in $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ indicated by Eq. (33) and the assumption that $|\underset{\sim}{k}| \gg |\underset{\sim}{K}|$, we need keep only the lowest order non-vanishing terms on expanding the functions in Eq. (37) in $\underset{\sim}{\rho}$ and $\underset{\sim}{K}$. With some little algebra, then, we find that

$$
\begin{aligned}
T_a = &- \underset{\sim}{C}_k\cdot\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x},\underset{\sim}{k}) + (\underset{\sim}{\alpha}\cdot\underset{\sim}{k}/v_k - 2\nu k^2)\, F(\underset{\sim}{x},\underset{\sim}{k}) \\
&- \underset{\sim}{U}\cdot\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x},\underset{\sim}{k}) + \left[\nabla_{\underset{\sim}{x}}(\underset{\sim}{k}\cdot\underset{\sim}{U})\right]\cdot\left[\nabla_{\underset{\sim}{k}} F(\underset{\sim}{x},\underset{\sim}{k})\right] \\
&- \left\{\nabla_{\underset{\sim}{x}}\cdot\left[(\underset{\sim}{k}\underset{\sim}{k}\cdot\underset{\sim}{U}/(2k^2)\right]\right\} F(\underset{\sim}{x},\underset{\sim}{k})
\end{aligned}
\tag{38}
$$

where $\underset{\sim}{C}_k \equiv \nabla_{\underset{\sim}{k}}\omega_k$, $\nabla_{\underset{\sim}{x}}$ is the gradient operator in the horizontal plane and $\nabla_{\underset{\sim}{k}}$ is the corresponding wave vector gradient.

The assumed form [Eq. (17)] of the fourth-order correlation function allows us to express T_b in Eq. (36) in terms of the spectral density function $F(\underset{\sim}{k},\underset{\sim}{x})$. A straightforward evaluation leads to the expression

$$
\begin{aligned}
T_b = \frac{i}{A_o^2} \sum_{\underset{\sim}{L}} \int d^2y\, F(\underset{\sim}{y},\underset{\sim}{L}) \int d^2r \sum_{\underset{\sim}{\rho},\underset{\sim}{q}} \exp\left[i\underset{\sim}{\rho}\cdot(\underset{\sim}{x}-\underset{\sim}{r}) + i\underset{\sim}{q}\cdot(\underset{\sim}{r}-\underset{\sim}{y})\right] \\
\times \left[\sigma^{\underset{\sim}{k}+\underset{\sim}{\rho}/2,\underset{\sim}{L}-\underset{\sim}{q}/2}_{\underset{\sim}{L}+\underset{\sim}{q}/2,\underset{\sim}{k}+\underset{\sim}{\rho}/2-\underset{\sim}{q}} F(\underset{\sim}{r},\underset{\sim}{k}-\underset{\sim}{q}/2) - \sigma^{\underset{\sim}{k}-\underset{\sim}{\rho}/2,\underset{\sim}{L}+\underset{\sim}{q}/2}_{\underset{\sim}{L}-\underset{\sim}{q}/2,\underset{\sim}{k}-\underset{\sim}{\rho}/2+\underset{\sim}{q}} F(\underset{\sim}{r},\underset{\sim}{k}+\underset{\sim}{q}/2)\right].
\end{aligned}
\tag{39}
$$

We expect $\underset{\sim}{\rho}$ and $\underset{\sim}{q}$ to have magnitudes of order W^{-1}, and thus to be very small compared with $\underset{\sim}{k}$ and $\underset{\sim}{L}$ in Eq. (39). We

can therefore expand $\mathcal{G}$ and F in terms of $\underset{\sim}{\rho}$ and $\underset{\sim}{q}$, keeping only first-order terms. We define $\underset{\sim}{\mathcal{D}}_1$ and $\underset{\sim}{\mathcal{D}}_2$ by the equations

$$\begin{aligned} \underset{\sim}{\mathcal{D}}_1 &\equiv \nabla_{\underset{\sim}{k}} \mathcal{G}^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}} \quad , \\ \underset{\sim}{q}\cdot\underset{\sim}{\mathcal{D}}_2 &\equiv \lim_{\underset{\sim}{q}\to 0} \left[\mathcal{G}^{\underset{\sim}{k},\underset{\sim}{L}+\underset{\sim}{q}/2}_{\underset{\sim}{L}-\underset{\sim}{q}/2,\underset{\sim}{k}+\underset{\sim}{q}} - \mathcal{G}^{\underset{\sim}{k},\underset{\sim}{L}-\underset{\sim}{q}/2}_{\underset{\sim}{L}+\underset{\sim}{q}/2,\underset{\sim}{k}-\underset{\sim}{q}} \right] \end{aligned} \tag{40}$$

and find that

$$\begin{aligned} T_b = \sum_{\underset{\sim}{L}} \Bigg\{ &-\mathcal{G}^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}} \left[\nabla_{\underset{\sim}{x}} F(\underset{\sim}{x},\underset{\sim}{L})\right] \cdot \left[\nabla_{\underset{\sim}{k}} F(\underset{\sim}{x},\underset{\sim}{k})\right] \\ &- F(\underset{\sim}{x},\underset{\sim}{k}) \left[\underset{\sim}{\mathcal{D}}_2 \cdot \nabla_{\underset{\sim}{x}} F(\underset{\sim}{x},\underset{\sim}{L})\right] + \underset{\sim}{\mathcal{D}}_1 \cdot \nabla_{\underset{\sim}{x}} \left[F(\underset{\sim}{x},\underset{\sim}{L})\ F(\underset{\sim}{x},\underset{\sim}{k})\right] \Bigg\} \quad . \end{aligned} \tag{41}$$

Using the forms of T_a and T_b given by Eqs. (38) and (41) in Eq. (36), we can express the time derivative of $F(\underset{\sim}{x},\underset{\sim}{k})$ in terms of F itself. A more convenient expression may be constructed by using $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ however. Employing Eqs. (29) and (30) and with a little re-arranging we obtain,

$$\left(\frac{\partial}{\partial t} + \frac{d\underset{\sim}{x}}{dt}\cdot\nabla_{\underset{\sim}{x}} + \frac{d\underset{\sim}{k}}{dt}\cdot\nabla_{\underset{\sim}{k}}\right) \Psi(\underset{\sim}{x},\underset{\sim}{k}) = S(\underset{\sim}{k})\ \Psi(\underset{\sim}{x},\underset{\sim}{k}) + (\underset{\sim}{\alpha}\cdot\underset{\sim}{k}/v_k - 2\nu k^2)\Psi(\underset{\sim}{x},\underset{\sim}{k}) . \tag{42}$$

In Eq. (42)

$$\begin{aligned} \frac{d\underset{\sim}{x}}{dt} &= \nabla_{\underset{\sim}{k}} \mathcal{H} \\ \frac{d\underset{\sim}{k}}{dt} &= -\nabla_{\underset{\sim}{x}} \mathcal{H} \end{aligned} \tag{43}$$

where

$$\mathcal{H} = \underset{\sim}{k}\cdot\underset{\sim}{U} + \omega_k - \int d^2L\, \mathcal{C}^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}}\, \Psi(\underset{\sim}{x},\underset{\sim}{L}) \tag{44}$$

and

$$S(\underset{\sim}{k}) = \left\{\nabla_{\underset{\sim}{x}}\cdot\left[-\underset{\sim}{k}\underset{\sim}{k}\cdot\underset{\sim}{U}/(2k^2) + \int d^2L(\underset{\sim}{\mathcal{B}}_1 - \underset{\sim}{\mathcal{B}}_2)\,\Psi(\underset{\sim}{x},\underset{\sim}{L})\right]\right\} . \tag{45}$$

We emphasize that the gradient operator $(\nabla_{\underset{\sim}{x}})$ in Eq. (45) does not act outside the curly brackets $\{\ldots\}$; i.e., does not act on $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ in Eq. (42).

Equations (43) and (44) have the form of the familiar ray equations of wave propagation in the approximation of geometric optics. With $\mathcal{H}$ having the form $\mathcal{H} = \underset{\sim}{k}\cdot\underset{\sim}{U} + \omega_K$ they have previously been used (Whitham, 1961) to study wave refraction (Kenyon, 1971) by ocean currents. The integral term in Eq. (44) represents the influence of nonlinear wave interactions on refraction and propagation. We shall describe some implications of this term in the following sections.

Were the right-hand side of Eq. (42) equal to zero, this equation could be integrated in terms of the "ray equations" (43) and (44). To do this, one first integrates Eq. (43) to find a parametrized set of solutions

$$\begin{aligned} \underset{\sim}{x} &= \underset{\sim}{x}(\underset{\sim}{x}_o, \underset{\sim}{k}_o, t) \\ \underset{\sim}{k} &= \underset{\sim}{k}(\underset{\sim}{x}_o, \underset{\sim}{k}_o, t) \end{aligned} \tag{46}$$

with the boundary conditions

$$\underset{\sim}{x} = \underset{\sim}{x}_o, \quad \underset{\sim}{k} = \underset{\sim}{k}_o \quad \text{at } t = 0 \quad .$$

At time t = 0, let us assume that

$$\Psi = \Psi_o(\underset{\sim}{x},\underset{\sim}{k}) \quad .$$

Then, at time t

$$\Psi(\underset{\sim}{x},\underset{\sim}{k}) = \Psi_o\left[\underset{\sim}{x}_o(\underset{\sim}{x},\underset{\sim}{k},t),\ \underset{\sim}{k}_o(\underset{\sim}{x},\underset{\sim}{k},t)\right] \quad , \tag{47}$$

where Eq. (46) has been inverted to express $\underset{\sim}{x}_o$ and $\underset{\sim}{k}_o$ as functions of $\underset{\sim}{x}$, $\underset{\sim}{k}$, and t.

The second term on the right in Eq. (42) represents, in a fairly obvious way, the implications of our modelling of wind and damping forces. The first term in the function $S(\underset{\sim}{k})$ can be rewritten in tensor notation as

$$-\sum_{i,j=1}^{2} \tfrac{1}{2}\left[\Psi(\underset{\sim}{x},\underset{\sim}{k})\ k_i k_j/(2k^2)\right] \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \quad .$$

This is seen to correspond to the radiation stress term introduced by Longuet-Higgins and Stewart (1960, 1961). The remaining portion of ΨS represents a kind of "stress" associated with the interaction of waves of wavenumber $\underset{\sim}{k}$ with the entire spectrum.

Had we kept higher order terms in Eq. (21) correspondingly higher order terms would have been obtained in Eq. (42). For example, fifth-order terms in Eq. (21) would have led us to third-order terms (similar to those found by Hasselmann, 1961, 1963) in Eq. (43). To what order one can continue and yet neglect fifth and higher order correlation functions is presently not evident.

V. The Phase Velocity of a "Test Wave"

The group velocity at wavenumber $\underset{\sim}{k}$ is obtained from Eqs. (43) and (44) as

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{U} + \underset{\sim}{c}_{\underset{\sim}{k}} - \int d^2L\, \mathcal{L}_1\, \Psi(\underset{\sim}{x},\underset{\sim}{L}) \quad . \tag{48}$$

The first two terms are obvious. The third term represents the influence of the nonlinear wave interactions. Since $\Psi(\underset{\sim}{x},\underset{\sim}{L})$ will in general be asymmetric due to the influence of wind and/or obstructions, the group velocity can have a component not parallel to $\underset{\sim}{k}$.

Some insight can be obtained into Eq. (48) by considering a "test wave" interacting with a spectrum of ocean waves in a uniform ocean. We imagine the test wave to be mechanically generated with identical characteristics for each of a sequence of observations. Thus, we write

$$a(\underset{\sim}{\ell}) = a_o(\underset{\sim}{\ell}) + a'(\underset{\sim}{k})\, \delta_{\underset{\sim}{k}-\underset{\sim}{\ell}} \quad , \tag{49}$$

where a_o is a random variable describing the ambient sea and a' represents the small amplitude "test wave". We substitute (49) into Eq. (21), neglect the prescribed current, wind and viscosity terms, and obtain a linear equation for $a'(\underset{\sim}{k})$. Because we have assumed a uniform ocean,

$$\langle a_o(\underset{\sim}{\ell})\, a_o^*(\underset{\sim}{p})\rangle = \delta_{\underset{\sim}{\ell}-\underset{\sim}{p}} \langle |a_o(\underset{\sim}{\ell})|^2\rangle \quad .$$

Integration with respect to time then gives us the angular frequency

$$\bar{\omega}_{\underset{\sim}{k}} = \omega_k - \int d^2L\, \mathcal{C}^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}}\, \Psi(\underset{\sim}{L}) \quad , \tag{50}$$

where we have indicated no $\underset{\sim}{x}$-dependence of Ψ. The phase velocity is $\bar{\omega}_{\underset{\sim}{k}}/k$ and the group velocity deduced from Eq. (50) is in agreement with that of Eq. (48).

To illustrate the implications of Eq. (50) we consider the spectrum of Tyler, et al. (1974), which is based on a representation proposed by Longuet-Higgins, Cartwright, and Smith (1963). This is

$$\Psi(\underset{\sim}{L}) \cong (0.4\times10^{-2}/L^4)\left[G(\beta)/N\right] \quad , \quad \text{for } k_o < L < k_\Gamma \quad ,$$

$$= 0 \quad \text{for } L < k_o \quad \text{or} \quad L > k_\Gamma \quad . \tag{51}$$

Here the angular variation of the spectrum is given by

$$G(\beta) = \alpha + (1-\alpha)\cos^{s(L)}(\beta/2)$$

and

$$N = \int_{-\pi}^{\pi} G(\beta)\, d\beta \quad .$$

In these equations α is very small ($\sim 10^{-2}$), and k_o and k_Γ are the respective long and short wavelength cut-offs of the spectrum, and β is the angle between $\underset{\sim}{L}$ and the wind direction. Finally, $s(L)$ is a function of wavenumber which is near unity for short wavelengths and becomes quite large compared with unity near $L = k_o$.

We shall evaluate Eq. (50) for wavelengths shorter than the cut-off, or

$$k >> k_o \quad . \tag{52}$$

In this case the principal contribution to the integral in Eq. (50) comes from L-values near $L = k_o$ and a simple analytic evaluation is possible.

The coefficient $\sigma^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}}$ is obtained from Eq. (c.1) of Appendix C. For $k >> L$, this is

$$\sigma^{\underset{\sim}{k},\underset{\sim}{L}}_{\underset{\sim}{L},\underset{\sim}{k}} \cong -\left(\frac{7}{4}\right) L\, v_L \underset{\sim}{k}\cdot\underset{\sim}{L} \quad . \tag{53}$$

On evaluating the integral we find that

$$\bar{\omega}_{\underset{\sim}{k}} \cong \omega_k \left[1 + 1.4\times 10^{-2} \cos\beta\cdot (k/k_o)^{\frac{1}{2}}\right] \quad , \tag{54}$$

where β is the angle between $\underset{\sim}{k}$ and the wind direction. The group velocity obtained from Eq. (54) is

$$\nabla_{\underset{\sim}{k}}\, \bar{\omega}_{\underset{\sim}{k}} = \underset{\sim}{C}_{\underset{\sim}{k}} + \hat{\underset{\sim}{W}} \left[1.4\times10^{-2} \cos\beta (g/k_o)^{\frac{1}{2}} \right] , \quad (55)$$

where $\underset{\sim}{W}$ is a unit vector parallel to the wind direction.

VI. Wave Shadowing by an Island

During the series of experiments reported in Tyler, et al. (1974), the "shadow" of an island for receding waves was observed*. At sufficiently large distances from the island this shadow is absent. There are evidently several possible causes for the filling in of the spectrum away from the island. One of these is nonlinear wave interactions, which we now discuss as an application of Eq. (42).

We calculate the filling in of the spectrum of wavenumbers $\underset{\sim}{k}$ directed away from the island and in its shadow. That is, $\Psi(\underset{\sim}{x},\underset{\sim}{k})$ will be very small where effective shadowing occurs. On the other hand, we assume that $\Psi(\underset{\sim}{x},\underset{\sim}{L}) = \Psi(\underset{\sim}{L})$ will not have much $\underset{\sim}{x}$-dependence for those waves $\underset{\sim}{L}$ which have "missed" the island. If the shadowing angle is small, we can take (we now suppose that $\underset{\sim}{U} = 0$ and the effects of wind and viscosity can be neglected)

$$S \cong 0, \quad \frac{dk}{dt} \cong 0$$

in Eq. (42). Equation (48) gives the group velocity with which waves of wavenumber $\underset{\sim}{k}$ propagate into the shadow

$$\frac{dx}{dt} \equiv \underset{\sim}{C}(\underset{\sim}{k}) = \underset{\sim}{C}_{\underset{\sim}{k}} - \int d^2L\, \mathcal{B}_1\, \Psi(\underset{\sim}{L}) \quad . \tag{56}$$

*We are indebted to Professor Walter Munk for describing these observations to us prior to publication.

If this were a time-dependent problem, with a sharply outlined shadow at t = 0, say $\Psi = \Psi_o(\underset{\sim}{x},\underset{\sim}{k})$, then at time t Eq. (47) would imply that

$$\Psi(\underset{\sim}{x},\underset{\sim}{k}) = \Psi_o(\underset{\sim}{x} - \underset{\sim}{C}(\underset{\sim}{k})t,\ \underset{\sim}{k}) \quad . \tag{57}$$

The expression (55) would lead us to expect a triangular shadow of half angle

$$\theta_i \cong 1.4\times10^{-2} \sin(2\beta)\ (k/k_o)^{\frac{1}{2}} \quad . \tag{58}$$

When waves travelling parallel to the wind are shadowed by the island, then the filling in of the spectrum will be modified. Should a significant portion of the spectrum be in the island shadow, then $\Psi(\underset{\sim}{L})$ in Eq. (56) must be appropriately modified.

VII. Other Correlation Functions

The correlation function $\langle a(\underset{\sim}{q})\ a(\underset{\sim}{k})\rangle$ is easily obtained in terms of F, or equivalently, $\langle a(\ell)\ a^*(p)\rangle$. Using Eq. (21) (and ignoring T_W and T_U), we obtain

$$\left[\frac{d}{dt} + i(\omega_k+\omega_q)\right]\langle a(\underset{\sim}{q})\ a(\underset{\sim}{k})\rangle = \sum_{\underset{\sim}{\ell},\underset{\sim}{p},\underset{\sim}{n}} \Big[\delta_{\underset{\sim}{k}+\underset{\sim}{n}+\underset{\sim}{p}-\underset{\sim}{\ell}}\ C_{\underset{\sim}{\ell}}^{\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{n}}$$

$$\times \langle a(\underset{\sim}{q})\ a(\underset{\sim}{\ell})\ a^*(\underset{\sim}{p})\ a^*(\underset{\sim}{n})\rangle$$

$$+ \delta_{\underset{\sim}{q}+\underset{\sim}{n}+\underset{\sim}{p}-\underset{\sim}{\ell}}\ C_{\underset{\sim}{\ell}}^{\underset{\sim}{q},\underset{\sim}{p},\underset{\sim}{n}} \langle a(\underset{\sim}{k})\ a(\underset{\sim}{\ell})\ a^*(\underset{\sim}{p})\ a^*(\underset{\sim}{n})\rangle\Big]$$

$$\equiv T(\underset{\sim}{k},\underset{\sim}{q}) \quad . \tag{59}$$

Thus

$$\langle a(\underset{\sim}{q})\ a(\underset{\sim}{k})\rangle \cong -i(\omega_k+\omega_q)^{-1}\, T(\underset{\sim}{k},\underset{\sim}{q}) \quad . \tag{60}$$

Use of Eq. (17) then permits explicit evaluation of this quantity.

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APPENDIX A: The Interaction of Surface Waves with a Current

In this Appendix we obtain the terms in Eq. (11) which represent the interaction of surface waves with the prescribed current $\underset{\sim}{U}$. Bernoulli's equation and the kinematic boundary condition at the surface are, respectively,

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} (\nabla \Phi)^2 + gh = 0 \quad ,$$

$$\frac{\partial h}{\partial t} + (\nabla_s \Phi) \cdot (\nabla_s h) = \frac{\partial \Phi}{\partial z} \quad , \tag{A.1}$$

where both equations are evaluated on the surface

$$z = h(\underset{\sim}{x}, t) \equiv H + \zeta \quad . \tag{A.2}$$

If we extract the long wavelength, low frequency part of Eq. (A.1) in linearized form, there results

$$\frac{\partial H}{\partial t} = \frac{\partial \hat{\Phi}}{\partial t} \quad ,$$

$$\frac{\partial \hat{\Phi}}{\partial t} + gH = 0 \quad , \tag{A.3}$$

both evaluated on the surface z = 0.

The current associated with $\hat{\Phi}$ is

$$\hat{\underset{\sim}{U}} = \nabla\hat{\Phi} = \hat{U}_z \hat{\underset{\sim}{k}} + \underset{\sim}{U} \quad ,$$

where $\underset{\sim}{U}$ is given by Eq. (2). We have assumed that

$$|\hat{U}_z| << |\underset{\sim}{U}| \quad . \tag{A.4}$$

Our conditions also imply that

$$|\nabla_s \ell n\ H| << |\nabla_s \ell n\ \zeta| \quad , \tag{A.5}$$

or that the surface curvature due to the current is much less than that due to the surface waves.

To extract the high frequency linearized terms from (A.1) we note that

$$\hat{\Phi}\Big|_{z=h} \cong \hat{\Phi}\Big|_{z=H} + \zeta \frac{\partial\hat{\Phi}}{\partial z}\Big|_{z=H} \quad ,$$

$$\frac{\partial\hat{\Phi}}{\partial z}\Big|_{z=h} \cong \frac{\partial\hat{\Phi}}{\partial z}\Big|_{z=H} + \zeta \frac{\partial^2\hat{\Phi}}{\partial z^2}\Big|_{z=H}$$

$$= \frac{\partial\hat{\Phi}}{\partial z}\Big|_{z=H} - \zeta\nabla_s\cdot\underset{\sim}{U} \quad ,$$

since $\nabla^2\hat{\Phi} = 0$.

Thus, the high frequency part of Eq. (A.1), linear in the surface wave amplitudes, and evaluated on the surface $z = H$, is

$$\frac{\partial\phi}{\partial t} + \hat{\underset{\sim}{U}}\cdot\nabla\phi + \zeta\frac{\partial U_z}{\partial t} + g\zeta = 0 \quad ,$$

$$\frac{\partial\zeta}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\zeta + (\nabla_s\phi)\cdot(\nabla_s H) + \zeta(\nabla_s\cdot\underset{\sim}{U}) = \frac{\partial\phi}{\partial z} \quad . \qquad \text{(A.6)}$$

Equation (A.6) is to be evaluated by replacing $\phi(\underset{\sim}{x},z,t)$ by

$$\phi(\underset{\sim}{x},H,t) \equiv \phi_H(\underset{\sim}{x},t) \qquad \text{(A.7)}$$

after the indicated differentiations are performed. Following a treatment of Milder (1973), we re-express Eq. (A.6) in terms of ϕ_H . For example,

$$\left(\frac{\partial}{\partial t} + \hat{\underset{\sim}{U}}\cdot\nabla\right)\phi\bigg|_{z=H} = \left(\frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\right)\phi_H \quad . \qquad \text{(A.8)}$$

This lets us replace the first of Eqs. (A.6) by

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\right)\phi_H + g_{eff}\,\zeta = 0 \quad , \qquad \text{(A.9)}$$

$$g_{eff} = g + \frac{\partial U_z}{\partial t} \quad . \qquad \text{(A.10)}$$

The unit normal $\hat{\underset{\sim}{n}}$ to the surface is, to first order in H,

$$\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{k}} - \nabla_s H \quad . \tag{A.11}$$

Thus, the second of Eqs. (A.6) is equivalent to the equation

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\right) \zeta + \zeta(\nabla_s \cdot \underset{\sim}{U}) = \hat{n}\cdot\nabla\phi \qquad \text{on } z = H \quad . \tag{A.12}$$

To express this in terms of ϕ_H, we use the relation $\nabla^2\phi = 0$ to write

$$\hat{n}\cdot\nabla\phi\Big|_{z=H} = ⊕\, \phi_H + \mathcal{O}\left(\frac{\partial^2 H}{\partial x^2}\,\phi_H\right) \quad , \tag{A.13}$$

and

$$⊕ \equiv \sqrt{-\nabla_s^2} \quad . \tag{A.14}$$

The conditions (A.4) and (A.5) finally give us the equations

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\right) \phi_H + g\zeta = 0 \quad ,$$

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{U}\cdot\nabla_s\right) \zeta + \zeta\, \nabla_s \cdot \underset{\sim}{U} = ⊕\, \phi_H \quad . \tag{A.15}$$

On writing

$$\zeta = i\,(Z - Z^{*})/2$$

$$\phi_H = (g/\oplus)^{\frac{1}{2}}\,(Z + Z^{*})/2 \tag{A.16}$$

and using the Fourier expansion [Eq. (7)], we obtain the required terms in Eq. (11).

APPENDIX B: The Wave-Wave Interaction

In this Appendix we show how to obtain the terms T_2 and T_3 in Eq. (11).

Extracting from Eqs. (A.1) the part which pertains to surface gravity waves, we obtain the equations

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}(\nabla\phi)^2 + g\zeta = 0 \quad , \qquad z = \zeta$$

$$\frac{\partial\zeta}{\partial t} + (\nabla_s\phi)\cdot(\nabla_s\zeta) = \frac{\partial\phi}{\partial z} \quad , \qquad z = \zeta \quad . \qquad \text{(B.1)}$$

We first re-express these equations in terms of $\phi(\underset{\sim}{x},z,t)$ evaluated on the surface $z = \zeta(\underset{\sim}{x},t)$; that is,

$$\phi_s(\underset{\sim}{x},t) \equiv \phi\left[\underset{\sim}{x},\ \zeta(\underset{\sim}{x},t),\ t\right] \quad . \qquad \text{(B.2)}$$

Then, we define

$$W(\underset{\sim}{x},t) \equiv \left.\frac{\partial\phi}{\partial z}\right|_{z=\zeta} \qquad \text{(B.3)}$$

and re-write Eqs. (B.1) as

$$\frac{\partial\phi_s}{\partial t} - W\frac{\partial\zeta}{\partial t} + \tfrac{1}{2}(\nabla_s\phi_s - W\nabla_s\zeta)^2 + \tfrac{1}{2}W^2 + g\zeta = 0$$

$$\frac{\partial\zeta}{\partial t} + (\nabla_s\phi_s - W\nabla_s\zeta)\cdot\nabla_s\zeta = W \quad . \qquad \text{(B.4)}$$

It remains to express W in terms of ϕ_s , which is a special and rather simple application of potential theory with a Dirichlet boundary condition (Jackson, 1962). This is easily done by first expressing both ϕ_s and W as Taylor series in ζ about the plane z = 0. Then W can be expressed in terms of ϕ_s by successive substitution. The result is

$$W = (\circledast\, \phi_s) - \left[\circledast(\zeta \circledast \phi_s) - (\zeta \circledast^2 \phi_s)\right] + \left\{\circledast\left[\zeta \circledast(\zeta \circledast \phi_s)\right] - \zeta\left[\circledast^2(\zeta \circledast \phi_s)\right]\right\}$$

$$- \tfrac{1}{2}\left\{\circledast\left[\zeta^2(\circledast^2 \phi_s)\right] - \zeta^2(\circledast^3 \phi_s)\right\} \quad . \qquad \text{(B.5)}$$

The term $\frac{\partial \zeta}{\partial t}$ can be eliminated from the first of Eqs. (B.4) using the second of these equations and W eliminated from both using Eq. (B.5). Finally, a first-order equation for

$$Z = -i\zeta + V_x^{-1}\, \phi_s \qquad \text{(B.6)}$$

can be obtained by differentiation with respect to time and substituting from Eqs. (B.4). The Fourier expansion [Eq. (7)] then gives us the terms T_2 and T_3 of Eq. (11).

The coefficients in T_2 [Eq. (15)] are

$$\Gamma^{\underset{\sim}{k}}_{\underset{\sim}{\ell},\underset{\sim}{p}} = \frac{1}{8}\left[(V_\ell V_p/V_k)(\ell p+\underset{\sim}{\ell}\cdot\underset{\sim}{p}) - V_p(kp-\underset{\sim}{k}\cdot\underset{\sim}{p}) - V_\ell(k\ell-\underset{\sim}{k}\cdot\underset{\sim}{\ell})\right] ,$$

$$\Gamma^{\underset{\sim}{k},\underset{\sim}{p}}_{\underset{\sim}{\ell}} = \frac{1}{4}\left[(V_\ell V_p/V_k)(\ell p-\underset{\sim}{\ell}\cdot\underset{\sim}{p}) - V_p(kp+\underset{\sim}{k}\cdot\underset{\sim}{p}) + V_\ell(k\ell-\underset{\sim}{k}\cdot\underset{\sim}{\ell})\right] ,$$

$$\Gamma^{\underset{\sim}{k},\underset{\sim}{\ell},\underset{\sim}{p}} = \frac{1}{8}\left[(V_\ell V_p/V_k)(\ell p+\underset{\sim}{\ell}\cdot\underset{\sim}{p}) + V_p(kp+\underset{\sim}{k}\cdot\underset{\sim}{p}) + V_\ell(k\ell+\underset{\sim}{k}\cdot\underset{\sim}{\ell})\right] . \quad (B.7)$$

The first coefficient in Eq. (16) (the only one required in this paper) is

$$\begin{aligned}\Gamma^{\underset{\sim}{k},\underset{\sim}{n}}_{\underset{\sim}{\ell},\underset{\sim}{p}} = \frac{1}{4}\Big[&(\omega_n-\omega_p)\ |\underset{\sim}{p}-\underset{\sim}{n}|\ (k-|\underset{\sim}{k}-\underset{\sim}{\ell}|) + (\omega_n-\omega_\ell)\ |\underset{\sim}{\ell}-\underset{\sim}{n}|\ (k-|\underset{\sim}{k}-\underset{\sim}{p}|)\\ &- (\omega_\ell+\omega_p)\ |\underset{\sim}{\ell}+\underset{\sim}{p}|\ (k-|\underset{\sim}{k}+\underset{\sim}{n}|) + \omega_p p(k-p) + \omega_\ell \ell(k-\ell) - \omega_n n(k-n)\\ &- \omega_\ell\ \underset{\sim}{p}\cdot\underset{\sim}{n} - \omega_p\ \underset{\sim}{\ell}\cdot\underset{\sim}{n} - 2\omega_n\ \underset{\sim}{\ell}\cdot\underset{\sim}{p} + (\omega_\ell\omega_n/\omega_k)k(n-|\underset{\sim}{n}-\underset{\sim}{p}|+\ell-|\underset{\sim}{\ell}+\underset{\sim}{p}|)\\ &+ (\omega_p\omega_n/\omega_k)k(n-|\underset{\sim}{n}-\underset{\sim}{\ell}|+p-|\underset{\sim}{p}+\underset{\sim}{\ell}|) - (\omega_\ell\omega_p/\omega_k)k(p-|\underset{\sim}{p}-\underset{\sim}{n}|+\ell-|\underset{\sim}{\ell}-\underset{\sim}{n}|\Big] .\end{aligned} \quad (B.8)$$

Since the condition (32) has been used in our derivation of Eq. (42), we must restrict ourselves to wavelengths small compared to the length parameter W. To do this, we suppose the coefficients (B.7) and (B.8) vanish if any of their wave-number arguments violate the condition (32).

APPENDIX C: <u>The Coefficients in Equation (25)</u>

For reference we quote the form of the coefficients of the a's in Eq. (25):

$$
\begin{aligned}
C^{\mathbf{k},\mathbf{n}}_{\boldsymbol{\ell},\mathbf{p}} = \Gamma^{\mathbf{k},\mathbf{n}}_{\boldsymbol{\ell},\mathbf{p}} + 4 \Bigg[& \frac{\Gamma^{\mathbf{k}}_{\boldsymbol{\ell},\mathbf{p}-\mathbf{n}} \, \Gamma^{\mathbf{p}-\mathbf{n},\mathbf{n}}_{\mathbf{p}}}{\omega_{p} - \omega_{n} - \omega_{|\mathbf{p}-\mathbf{n}|}} + \frac{\Gamma^{\mathbf{k}}_{\mathbf{p},\boldsymbol{\ell}-\mathbf{n}} \, \Gamma^{\boldsymbol{\ell}-\mathbf{n},\mathbf{n}}_{\boldsymbol{\ell}}}{\omega_{\ell} - \omega_{n} - \omega_{|\boldsymbol{\ell}-\mathbf{n}|}} \\
& - \tfrac{1}{2} \frac{\Gamma^{\mathbf{k},\mathbf{n}-\mathbf{p}}_{\boldsymbol{\ell}} \, \Gamma^{\mathbf{n}-\mathbf{p},\mathbf{p}}_{\mathbf{n}}}{\omega_{n} - \omega_{p} - \omega_{|\mathbf{p}-\mathbf{n}|}} - \tfrac{1}{2} \frac{\Gamma^{\mathbf{k},\mathbf{n}-\boldsymbol{\ell}}_{\mathbf{p}} \, \Gamma^{\mathbf{n}-\boldsymbol{\ell},\boldsymbol{\ell}}_{\mathbf{n}}}{\omega_{n} - \omega_{\ell} - \omega_{|\boldsymbol{\ell}-\mathbf{n}|}} \\
& - \frac{\Gamma^{\mathbf{k},\mathbf{n}}_{\boldsymbol{\ell}+\mathbf{p}} \, \Gamma^{\boldsymbol{\ell}+\mathbf{p}}_{\boldsymbol{\ell},\mathbf{p}}}{\omega_{|\mathbf{p}+\boldsymbol{\ell}|} - \omega_{\ell} - \omega_{p}} + 2 \frac{\Gamma^{\mathbf{k},\mathbf{n},-(\boldsymbol{\ell}+\mathbf{p})} \, \Gamma^{-(\boldsymbol{\ell}+\mathbf{p}),\boldsymbol{\ell},\mathbf{p}}}{\omega_{\ell} + \omega_{p} + \omega_{|\boldsymbol{\ell}+\mathbf{p}|}} \Bigg] . \qquad \text{(C.1)}
\end{aligned}
$$

For the evaluation of the coefficients (40) certain of the terms in (C.1) appear to be singular, corresponding to the resonant excitation of arbitrarily long wavelengths. In accordance with the discussion following Eq. (B.8), these terms are to be dropped, corresponding to the assumed vanishing of the Γ-coefficients.

For the evaluation of Eq. (53), one should note the sequence of cancellations of the terms with powers of k greater than the first.

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